NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1230

EFFECT OF THE ACCELERATION OF ELONGATED BODIES OF REVOLUTION UPON THE RESISTANCE IN A COMPRESSIBLE FLOW

By F. I. Frankl

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EFFECT OF THE ACCELERATION OF ELONGATED BODIES OF REVOLUTION UPON THE RESISTANCE IN A COMPRESSIBLE GAS*

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The problem of the motion of an elongated body of revolution in an incompressible fluid may, as is known, be solved approximately with the aid of the distribution of sources along the axis of the body. In determining the velocity field, the question of whether the body moves uniformly or with an acceleration is no factor in the problem. The presence of acceleration must be taken into account in determining the pressures acting on the body. The resistance of the body arising from the accelerated motion may be computed either directly on the basis of these pressures or with the aid of the so-called associated masses (inertia coefficients). A different condition holds in the case of the motion of bodies in a compressible gas. In this case the finite velocity of sound must be taken into account. If it is assumed that the body produces in the flow only small disturbances, the velocity potential φ satisfies the wave equation:

$$a^2\Delta\varphi = \frac{\partial^2\varphi}{\partial t^2} \tag{1}$$

where a is the velocity of sound. The method of sources still remains applicable but in computing the effect of the sources it is necessary to make use of the retarded potential.

We introduce two systems of coordinates, one a fixed system xr, the other moving with the body ξr, where x and ξ are, respectively, the coordinates along the axis and r is the distance from the axis. The coordinate ξ is computed from the nose of the body.

Let $r = \bar{r}(\xi)$ be the equation of the body of revolution $(0 < \xi < l)$. The position of the body is characterized by the inequalities $-f(t) < x < l - f(t)$, where $f(t)$ is a given function of the time characterizing the motion of the body. The coordinates x and ξ are connected by the equation $x = \xi - f(t)$.

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*"O Vliianii uskoreniya na soprotivlenie pri dvizhenii prodolgovatykh tel vrashcheniya v gazakh." Prikladnaya Matematika i Mekhanika, Vol. 10, no. 4, 1946, pp. 521-524.

Let $q = q(x',t')$ be the value of the linear density of the source at the point x' at the instant of time t' and $R = \sqrt{(x - x')^2 + r^2}$ be the distance between the source and the point considered. The potential of the disturbance velocities is then

$$\varphi(x,t) = -\frac{1}{4\pi}\int \frac{1}{R} q \left(x', t - \frac{R}{a}\right) dx'$$

where the integral is taken between the limits determined by the inequalities

$$-f\left(t - \frac{R}{a}\right) < x' < l - f\left(t - \frac{R}{a}\right)$$

In order to establish the range of integration, we must consider in the plane $x't'$ a strip $-f(t') < x' < l - f(t')$ between the lines of motion of the front and rear points of the body (fig. 1) and the part of the semihyperbola in this strip.

$$t' = t - \frac{R}{a} = t - \frac{1}{a}\sqrt{(x' - x)^2 + R^2_{\;r^2}}$$

The abscissas of the part of the semihyperbola considered form the range of integration (fig. 1). As may be seen from the figure, the range of integration may consist of one or several segments. The ends of the corresponding arcs of the hyperbola may lie either on the line of motion of the nose or on the line of motion of the base. In particular, in the motion of the body with supersonic velocity it is possible that within the strip considered there lies an entire arc of the hyperbola, both ends of which lie on the line of motion of the nose (as will be the case, for example, for a uniform supersonic motion or for a motion approximating this type).

The function $q(x',t')$ must be determined from the integral equation expressing the fact that the normal component of the velocity of the gas on the surface of the body is equal to the normal component of the motion of the body:

$$\frac{\partial\varphi}{\partial n} = -\frac{1}{4\pi}\frac{\partial}{\partial n}\int\left[\frac{1}{R} q\left(x', t - \frac{R}{a}\right)\right]_{r=\bar{r}} dx' = \frac{d\bar{r}/d\xi}{\sqrt{1 + \left[\bar{r}'(\xi)\right]^2}} f'(t) \qquad (2)$$

where $\partial/\partial n$ denotes differentiation along the outer normal and $\xi = x + f(t)$. In what follows we consider the derivative $\partial\bar{r}/\partial\xi$ as small and shall give an approximate solution of equation (2). For the assumption made, $\partial\varphi/\partial n$ may be replaced by $\partial\varphi/\partial r$. We assume for simplicity that the range of integration reduces to the segment (x_1,x_2). Equation (2) then assumes the form

$$-\frac{1}{4\pi}\frac{\partial}{\partial r}\int_{x_1}^{x_2}\left[\frac{1}{R}q\left(x',t-\frac{R}{a}\right)\right]_{r=\bar{r}}dx' = \frac{1}{4\pi}\left\{r\int_{x_1}^{x_2}\left(\frac{q}{R^3}+\frac{1}{aR^2}\frac{\partial q}{\partial t'}\right)_{r=\bar{r}}dx'\right.$$
$$\left.-\frac{1}{R_2}q\left(x_2,t-\frac{R_2}{a}\right)\frac{dx_2}{dr}+\frac{1}{R_1}q\left(x_1,t-\frac{R_1}{a}\right)\frac{\partial x_1}{\partial r}\right\} = f'(t)\frac{d\bar{r}}{d\xi} \qquad (3)$$

If $\bar{r}/l$ and $d\bar{r}/d\xi$ are considered small quantities of the first order and their higher degrees or derivatives are neglected, an approximate solution of equation (3) will be[1]

$$q(x,t) = 2\pi f'(t)\bar{r}(\xi)\frac{d\bar{r}}{d\xi} \qquad (4)$$

We shall show that the error in substituting this expression in equation (3) is a magnitude of the second order of smallness. For this purpose it is convenient to write ϵr^* for $\bar{r}$ where $r^* = r^*(\xi)$ is a variable quantity of the order of unity relative to l and ϵ is a small constant number. The degree of ϵ in each expression then shows its order of smallness. We consider the order of the terms of the left side of equation (3). The terms outside the integral on the left of equation (3) containing q as a factor for finite terms (in the general case) will be of the second order of smallness. In the same way the expression

$$\left|\frac{r}{a}\int_{x_1}^{x_2}\frac{\partial q}{\partial t'}\frac{dx'}{R^2}\right| < M\epsilon^2 r\int_{x_1}^{x_2}\frac{dx'}{R^2} < M\pi\epsilon^2 \qquad (5)$$

is of second-order smallness. We write the remaining term in the form

$$\frac{r}{4\pi}\int_{x_1}^{x_2}\frac{1}{R^3}q\left(x',t-\frac{R}{a}\right)dx' = \frac{r}{4\pi}q(x,t)\int_{x_1}^{x_2}\frac{dx'}{R^3}+\frac{r}{4\pi}\int_{x_1}^{x_2}\left[q(x',t)-q(x,t)\right]\frac{dx'}{R^3}$$
$$+\frac{r}{4\pi}\int_{x_1}^{x_2}\left[q\left(x',t-\frac{R}{a}\right)-q(x',t)\right]\frac{dx'}{R^3} \qquad (6)$$

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[1]This solution (4) was first used by Kármán (reference 2) in the case of a uniform motion (in particular, for obtaining the approximate value of the wave resistance).

As a result of the computation of the first integral of this expression we obtain

$$r\int_{x_1}^{x_2}\frac{dx'}{R^3} = \frac{1}{\epsilon r^*(\xi)}\left[2 + O(\epsilon^2)\right] \tag{7}$$

For the estimates of the differences in the brackets in the second and third integrals of expression (6) we have, respectively

$$q(x',t) - q(x,t) = O(\epsilon^2)R, \qquad q(x',t) - q(x',t - R/a) = O(\epsilon^2)R \tag{8}$$

Hence the estimates of integrals are of the form

$$O(\epsilon^2)r\int_{x_1}^{x_2}\frac{dx'}{R^2} \leqslant \pi O(\epsilon^2) \tag{9}$$

so that these integrals are likewise magnitudes of the second order of smallness. Thus

$$-\frac{1}{4\pi}\frac{\partial}{\partial r}\int_{x_1}^{x_2}\left[\frac{1}{R}q\left(x',t - \frac{R}{a}\right)\right]_{r=\overline{r}} dx' = f'(t)\,\frac{d\overline{r}}{d\xi} + O(\epsilon^2) \tag{10}$$

as was required to be proven.

The expression for the velocity potential after substituting in equation (4) becomes

$$\varphi(x,t) = -\frac{1}{2}\int\frac{1}{R}\,f'\left(t - \frac{R}{a}\right)\gamma(\xi')\,dx' \qquad \left(\gamma(\xi) = \overline{r}\,\frac{d\overline{r}}{d\xi}\right) \tag{11}$$

For computing the field of pressures we have the generalized formula of Bernoulli

$$\frac{p - \overline{p}}{\rho} = -\frac{w^2}{2} - \frac{\partial\varphi}{\partial t} \tag{12}$$

where p is the pressure at the given point, $\overline{p}$ the pressure in the undisturbed region, ρ the density and, w the modulus of the velocity. Neglecting magnitudes of the second order $w^2/2$ and taking account of the equations

$$\xi' = x' + f\left(t - \frac{R}{a}\right), \qquad \frac{\partial\xi'}{\partial t} = f'\left(t - \frac{R}{a}\right) \tag{13}$$

we obtain

$$\frac{p - \bar{p}}{\rho} = \frac{1}{2} \int_{x_1}^{x_2} \left\{ \left[f'\left(t - \frac{R}{a}\right) \right]^2 \gamma'(\xi') + f''\left(t - \frac{R}{a}\right) \gamma(\xi') \right\} \frac{dx'}{R} - \left[\frac{\gamma(\xi')}{2R} f'\left(t - \frac{R}{a}\right) \right]_{x'=x_2} \frac{\partial x_2}{\partial t} \tag{14}$$

In the particular case where the pressure is required on the surface of a projectile, the motion of which approximates to uniform supersonic motion, the point x_2 lies on the line of motion of the nose of the projectile and the last term on the right of equation (14) is equal to zero. We obtain

$$\frac{p - \bar{p}}{\rho} = \frac{1}{2} \int_{x_1}^{x_2} \left\{ \left[f'\left(t - \frac{R}{a}\right) \right]^2 \gamma'(\xi') + f''\left(t - \frac{R}{a}\right) \right\} \frac{dx'}{R} \tag{15}$$

where x_1 and x_2 are determined from the equation

$$x_{1,2} = f\left(t - \frac{1}{a}\sqrt{\left(x_{1,2} - x\right)^2 + \bar{r}^2}\right) \tag{16}$$

In the case of uniform motion $x = \xi' - vt + c$ and equations (15) and (16) become

$$\frac{p - \bar{p}}{\rho} = \frac{v^2}{2} \int_{x_1^*}^{x_2^*} \gamma'(\xi') \frac{dx'}{R}, \qquad x_{1.2}^* = -v\left(t - \frac{1}{a}\sqrt{(x_{1.2}^* - x)^2 + r^2}\right) + c \tag{17}$$

The additional pressure produced by the acceleration is therefore given by the equation

$$\delta p = \frac{\rho}{2}\left\{ \int_{x_1}^{x_2} f''\left(t - \frac{R}{a}\right)\gamma(\xi') \frac{dx'}{R} + \int_{x_1}^{x_2} \left\{ \left[f'\left(t - \frac{R}{a}\right) \right]^2 - \left[f'(t)\right]^2 \right\} \gamma'(\xi') \frac{dx'}{R} \right.$$
$$+ \left[f'(t)\right]^2 \int_{x}^{x_2} \left[\gamma'(\xi') - \gamma'(\xi^{*\prime})\right] \frac{dx'}{R} + \int_{x_1}^{x_1^*} \left[f(t)\right]^2 \gamma'(\xi') \frac{dx'}{R}$$
$$\left. + \int_{x_2^*}^{x_2} \left[f'(t)\right]^2 \gamma'(\xi') \frac{dx'}{R} \right\} \tag{18}$$

where $\xi^{*'}$ is determined from the equation

$$x' = \xi^{*'} - f'(t)t' + f'(t)t - f(t) \tag{19}$$

From the structure of equation (18) it is easily seen that the relative increase of the pressure arising from the acceleration has, for velocities comparable with the velocity of sound, an order of magnitude bl/v^2 where $b = f''(t)$ is the acceleration. Hence for rocket projectiles of the usual dimensions, for accelerations not higher than 1000 meters per second^2 and velocities comparable with the velocity of sound, the added pressures arising from the acceleration are negligibly small.

This is confirmed also by the following illustrative computation. The length of the war head portion of the shell is equal to $l = 0.25$ meter and the maximum radius $\bar{r} = 0.07$ meter. The generator of the head is an arc of a parabola which smoothly goes over into the cylindrical part so that it is given by the equation $\bar{r} = 0.56\xi - 1.12\xi^2$ (scale in meters). The motion is one of uniform acceleration with an acceleration of 1000 meters per second^2 so that the line of motion of the projectile $x = -5000t^2$ $(t > 0)$, $x = 0$ $(t < 0)$.

The pressure distribution over the war head is found at the instant of time $t = 0.5$ second such that the velocity is equal to $v = 500$ meters per second. The air density $\rho = 0.125$ kilograms^{-1}per second^2. There were obtained for the pressure distributions (with account taken of the acceleration) and the additional pressures δp produced by the acceleration the following values:

$$\xi = 0.05 \quad 0.10 \quad 0.15 \quad 0.20 \qquad \xi = 0.05 \quad 0.10 \quad 0.15 \quad 0.20 \quad \text{(meters)}$$

$$p - \bar{p} = 8550, 7448, 5970, 3588; \ \delta p = -5.0 \ -5.5 \ -11.0 \ -16.0 \quad \text{(kilograms per meter}^2\text{)}$$

The wave resistance and the corresponding added resistance were then obtained as

$$Q = 2\pi \int_0^l \bar{r} \frac{d\bar{r}}{d\xi} (p - \bar{p}) \, d\xi = 100, \qquad \delta Q = 2\pi \int_0^l \bar{r} \frac{d\bar{r}}{d\xi} \delta p \, d\xi = 0.13 \quad \text{(kilograms)} \tag{20}$$

The added pressures δp arising from the acceleration were computed from the approximate equation (18) which for the example given has the form

$$\delta p = \frac{\rho b}{2}\left\{\int_{x_1}^{x_2} \frac{\gamma(\xi')}{R}\,dx' - 2\frac{bt}{a}\int_{x_1}^{x_2} \gamma'(\xi')\,dx' + \left(\frac{bt}{a}\right)^2 \int_{x_1}^{x_2} \gamma''(\xi')\,R\,dx' + b\gamma'(0)\left(\frac{{t_2}^2\delta x_2}{R_2} - \frac{{t_1}^2\delta x_1}{R}\right)\right\} \tag{21}$$

where

$$\delta x_1 = x_1 - x_1^* \approx \frac{\frac{b}{2}\left(\frac{R_1}{a}\right)^2}{\frac{bt}{a}\,\frac{x_1 - x}{R_1} - 1}, \qquad \delta x_2 = x_2 - x_2^* \approx \frac{\frac{b}{2}\left(\frac{R_2}{a}\right)^2}{\frac{bt}{a}\,\frac{x - x_2}{R_2} - 1} \tag{22}$$

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REFERENCES

1. Lamb, H.: Hydrodynamics. Cambridge, 6th edition, 1932.

2. v. Kármán, Th.: Problem of Resistance in a Compressible Fluid. 1935.

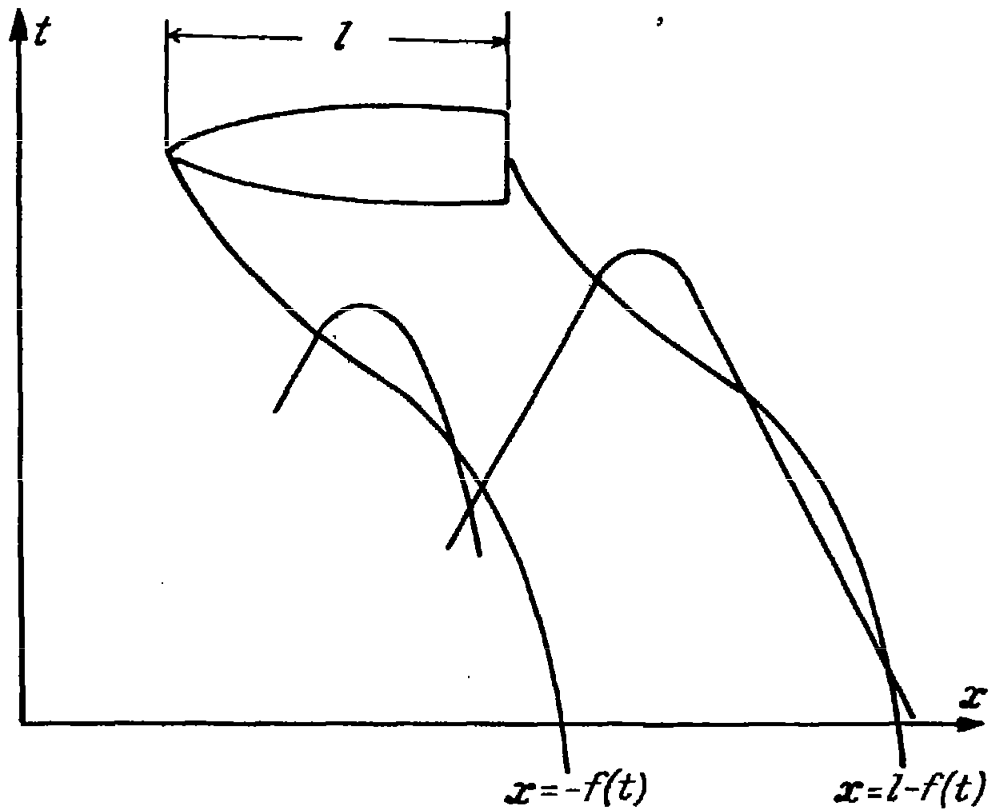

Figure 1.